ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Computational and Applied Mathematics 195 (2006) 312–325

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Block-based Thiele-like blending rational interpolation☆

Qian-Jin Zhao[a], Jieqing Tan[b,*]

[a]*School of Computer & Information, Hefei University of Technology, Hefei 230009, P. R. China*
[b]*Institute of Applied Mathematics, Hefei University of Technology, Hefei 230009, P. R. China*

Received 15 August 2004; received in revised form 15 March 2005

Abstract

Thiele-type continued fraction interpolation may be the favoured nonlinear interpolation in the sense that it is constructed by means of the inverse differences which can be calculated recursively and produce useful intermediate results. However, Thiele's interpolation is in fact a point-based interpolation by which we mean that a new interpolating continued fraction with one more degree of its numerator or denominator polynomial is obtained by adding a tail to the current one, or in other words, this can be reshaped simply by adding a new support point to the current set of support points once at a time. In this paper we introduce so-called block-based inverse differences to extend the point-based Thiele-type interpolation to the block-based Thiele-like blending rational interpolation. The construction process may be outlined as follows: first of all, divide the original set of support points into some subsets (blocks), then construct each block by using whatever interpolation means, linear or rational, and finally assemble these blocks by Thiele's method to shape the whole interpolation scheme. Clearly, many flexible rational interpolation schemes can be obtained in this case including the classical Thiele's continued fraction interpolation as its special case. As an extension, a bivariate analogy is also discussed and numerical examples are given to show the effectiveness of our method.

MSC: 41A20; 65D05

Keywords: Interpolation; Block-based inverse differences; Blending method

☆ Project supported by the National Natural Science Foundation of China under Grant No. 10171026, No. 60473114 and the Anhui Provincial Natural Science Foundation, China under Grant No. 03046102.
* Corresponding author.
E-mail address: jqtan@mail.hf.ah.cn (J. Tan).

0377-0427/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2005.03.089

1. Introduction

Denote by Π_n the set of all real or complex polynomials $p(x)$ with degree not exceeding n. Let $S_n = \{(x_i, f_i), i = 0, 1, \ldots, n\}$ be a set of support points, where the support abscissae x_i, $i = 0, 1, \ldots, n$, do not have to be distinct from one another; then an interpolating polynomial $P_n(x)$ in Π_n can be uniquely determined by S_n. Suppose the support ordinates f_i, $i = 0, 1, \ldots, n$, are the values of a given function $f(x)$ which is defined on the set $I(I \supset X_n)$, here $X_n = \{x_i, i = 0, 1, \ldots, n\}$; then Newton's polynomial $P_n(x)$ satisfying $P_n(x_i) = f(x_i)$, $i = 0, 1, \ldots, n$, is of the form [6]

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}),$$

where $f[x_0, x_1, \ldots, x_i]$ are the divided differences of $f(x)$ at support abscissae $x_0, x_1, \ldots, x_i$, which are defined by the recursion

$$\begin{aligned}
&f[x_i] = f(x_i),\\
&f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j},\\
&f[x_i, \ldots, x_j, x_k, x_l] = \frac{f[x_i, \ldots, x_j, x_l] - f[x_i, \ldots, x_j, x_k]}{x_l - x_k}.
\end{aligned}$$

We wish to mention that Newton interpolating polynomials have their nonlinear counterparts, the Thiele's interpolating continued fractions, which are constructed on the basis of inverse differences. Thiele's continued fraction interpolating the support points S_n possesses the following form:

$$R_n(x) = f(x_0) + \frac{x - x_0|}{|a_1} + \frac{x - x_1|}{|a_2} + \cdots + \frac{x - x_{n-1}|}{|a_n},$$

where for $i = 1, 2, \ldots, n$

$$a_i = \phi[x_0, x_1, \ldots, x_i]$$

is the inverse difference of $f(x)$ at $x_0, x_1, \ldots, x_i$, which can be computed recursively as follows:

$$\begin{aligned}
&\phi[x_i] = f(x_i), \ \ i = 0, 1, \ldots, n,\\
&\phi[x_i, x_j] = \frac{x_i - x_j}{f(x_i) - f(x_j)},\\
&\phi[x_i, x_j, x_k] = \frac{x_k - x_j}{\phi[x_i, x_k] - \phi[x_i, x_j]},\\
&\phi[x_i, \ldots, x_j, x_k, x_l] = \frac{x_l - x_k}{\phi[x_i, \ldots, x_j, x_l] - \phi[x_i, \ldots, x_j, x_k]}.
\end{aligned}$$

It is easy to verify that $R_n(x)$ is a rational function with degrees of numerator and denominator polynomials bounded by $[(n + 1)/2]$ and $[n/2]$, respectively, where $[x]$ denotes the greatest integer not exceeding x, and $R_n(x)$ satisfies

$$R_n(x_i) = f(x_i), \ \ i = 0, 1, \ldots, n.$$

The above two interpolants are purely linear or purely nonlinear, respectively. Blending rational interpolants may be better than the purely linear or nonlinear ones when the approximated functions have both linear and nonlinear properties simultaneously. One of the authors [9] established an extraordinary variety of rational interpolants by applying Neville's algorithm to continued fractions. One may say that the Thiele-type interpolation is a point-based interpolation since a new interpolating continued fraction with one more degree of its numerator or denominator polynomial is obtained by adding a new support point to the current set of support points once at a time. For interpolation by Thiele-type continued fractions, we refer to [1–3,5,7,8]. To obtain flexible blending rational interpolation, we try to extend the point-based interpolation to the block-based one. The idea may be outlined as follows: first divide the original set of support points into some subsets (blocks), then construct each block by using whatever interpolation means, linear or rational, and finally assemble these blocks by Thiele's method to shape the whole interpolation scheme.

2. Block blending interpolation

2.1. Basic idea

Suppose the set X_n is divided into $u+1$ subsets X_n^s $(s=0,1,\ldots,u)$:

$$X_n^s=\{x_{c_s},x_{c_s+1},\ldots,x_{d_s}\},\quad (s=0,1,\ldots,u).$$

The subsets may be achieved by reordering the interpolation points if necessary. Obviously, we have

$$\sum_{s=0}^{u}(d_s-c_s+1)=n+1.$$

Let us consider the following function with the Thiele-like form:

$$T(x)=I_0(x)+\frac{\omega_0(x)|}{|I_1(x)}+\cdots+\frac{\omega_{u-1}(x)|}{|I_u(x)}, \tag{1}$$

where the polynomials

$$\omega_s(x)=\prod_{i=c_s}^{d_s}(x-x_i),\quad s=0,1,\ldots,u-1, \tag{2}$$

and $I_s(x)$ $(s=0,1,\ldots,u)$ are polynomials or rational interpolating functions on the subsets X_n^s $(s=0,1,\ldots,u)$.

If the above $I_s(x)$ $(s=0,1,\ldots,u)$ are chosen so that

$$T(x_i)=f(x_i),\quad x_i\in X_n, \tag{3}$$

then $T(x)$ defined by (1) and (2) is called the block-based Thiele-like blending rational interpolant to $f(x)$.

2.2. Block-based inverse differences

Suppose $X_n \subset I \subset R$, and let $f(x)$ be a real function defined on I such that

$$f(x_i) = f_i, \quad i = 0, 1, \ldots, n. \tag{4}$$

We introduce the following notations:

$$f_i^0 = f_i, \quad i = 0, 1, \ldots, n \tag{5}$$

and define for $s = 1, 2, \ldots, u$

$$f_i^s = \frac{\omega_{s-1}(x_i)}{f_i^{s-1} - I_{s-1}(x_i)}, \quad i = c_s, c_s + 1, \ldots, n, \tag{6}$$

where $I_s(x)$ $(s = 0, 1, \ldots, u)$ are interpolating polynomials or rational interpolating functions on the subsets X_n^s $(s = 0, 1, \ldots, u)$, which satisfy

$$I_s(x_i) = f_i^s, \quad i = c_s, c_s + 1, \ldots, d_s; \quad s = 0, 1, \ldots, u. \tag{7}$$

If all f_i^s exist, then they are called the sth block-based inverse differences for the function $f(x)$.

Theorem 1. *Let*

$$T_u(x) = I_u(x) \tag{8}$$

and

$$T_s(x) = I_s(x) + \frac{\omega_s(x)\,|}{|\, I_{s+1}(x)} + \cdots + \frac{\omega_{u-1}(x)\,|}{|\, I_u(x)}, (s = 0, 1, \ldots, u-1). \tag{9}$$

If all the block-based inverse differences f_i^s $(i = c_s, c_s + 1, \ldots, d_s;\ s = 1, 2, \ldots, u)$ *in* (7) *exist and*

$$T_{s+1}(x_i) \neq 0, \quad (s = 0, 1, \ldots, u-1;\ i = c_s, c_s + 1, \ldots, d_s) \tag{10}$$

holds, then

$$T(x_i) = f_i, \quad i = 0, 1, \ldots, n.$$

Proof. Suppose $c_s \leqslant i \leqslant d_s$, $s = 0, 1, \ldots, u$. By (1), (5), (6), (7) and (10), we have

$$\begin{aligned} T(x_i) &= I_0(x_i) + \frac{\omega_0(x_i)|}{|I_1(x_i)} + \cdots + \frac{\omega_{s-1}(x_i)|}{|I_s(x_i)} \\ &= I_0(x_i) + \frac{\omega_0(x_i)|}{|I_1(x_i)} + \cdots + \frac{\omega_{s-1}(x_i)|}{|f_i^s} \\ &= I_0(x_i) + \frac{\omega_0(x_i)|}{|I_1(x_i)} + \cdots + \frac{\omega_{s-3}(x_i)|}{|I_{s-2}(x_i)} + \frac{\omega_{s-2}(x_i)|}{|f_i^{s-1}} \\ &= \cdots = f_i^0 = f_i. \end{aligned}$$

The proof is completed. □

2.3. Special cases

Block-based Thiele-like blending rational interpolation can be obtained by means of the above recursive algorithm and includes the following interesting special cases.

Case 1: If all the $I_s(x)$ $(s=0, 1, \ldots, u)$ are the interpolating polynomials $P_s(x)$ on the subsets X_n^s $(s=0, 1, \ldots, u)$, then we obtain

$$T(x) = P_0(x) + \frac{\omega_0(x)|}{|P_1(x)} + \cdots + \frac{\omega_{u-1}(x)|}{|P_u(x)}, \tag{11}$$

where $P_s(x)$ $(s=0, 1, \ldots, u)$ are the Newton interpolating polynomials on the subsets X_n^s $(s=0, 1, \ldots, u)$. In a particular case when $u = 0$, the unique subset is the whole set X_n, then the block-based Thiele-like blending rational interpolant degenerates into Newton interpolating polynomials on the whole set X_n.

Case 2: If all the $I_s(x)$ $(s = 0, 1, \ldots, u)$ are the Thiele-type interpolating continued fraction $R_s(x)$ on the subsets X_n^s $(s = 0, 1, \ldots, u)$, then the block-based Thiele-like blending rational interpolant becomes a kind of composite rational interpolant on the whole set X_n:

$$T(x) = R_0(x) + \frac{\omega_0(x)|}{|R_1(x)} + \cdots + \frac{\omega_{u-1}(x)|}{|R_u(x)}. \tag{12}$$

Especially, when $u = n$, each subset only contains one point, then all the block-based inverse differences become classical inverse differences and the block-based Thiele-like blending rational interpolant turns out to be the classical Thiele-type continued fraction interpolant on the whole set X_n.

If the whole set X_n is a unique subset, i.e., $u = 0$, then we have

$$T(x) = R_0(x), \tag{13}$$

where $R_0(x)$ is the Thiele-type interpolating continued fraction on the whole set X_n, and the block-based Thiele-like blending rational interpolant degenerates into the classical Thiele-type interpolating continued fraction on the whole set X_n.

2.4. Error estimation

We now turn to a discussion of the error in the approximation of a function $f(x)$ by its block-based Thiele-like blending rational interpolants.

Theorem 2. *Suppose $[a, b]$ is the smallest interval containing $X_n = \{x_0, x_1, \ldots, x_n\}$ and $f(x)$ is differentiable in $[a, b]$ up to $(n+1)$ times; let*

$$T(x) = I_0(x) + \frac{\omega_0(x)|}{|I_1(x)} + \cdots + \frac{\omega_{u-1}(x)|}{|I_u(x)} = \frac{P(x)}{Q(x)}, \tag{14}$$

then for each $x \in [a, b]$ there exists a point $\xi \in (a, b)$ such that

$$f(x) - T(x) = \frac{\omega(x)}{Q(x)} \cdot \frac{[f(x)Q(x) - P(x)]^{(n+1)}_{x=\xi}}{(n+1)!}, \tag{15}$$

where $\omega(x) = \prod_{i=0}^{n}(x - x_i)$.

Proof. Let $E(x) = f(x)Q(x) - P(x)$; then it follows from Theorem 1 and (14)

$$E(x_i) = 0, \quad (i = 0, 1, \ldots, n);$$

making use of the Newton interpolation formula (see [6,11]), we have

$$\begin{aligned} E(x) &= \sum_{i=0}^{n} E[x_0, x_1, \ldots, x_i](x - x_0)\cdots(x - x_{i-1}) \\ &\quad + (x - x_0)\cdots(x - x_n) \cdot \frac{E^{(n+1)}(\xi)}{(n+1)!} \\ &= \frac{\omega(x)E^{(n+1)}(\xi)}{(n+1)!}, \end{aligned}$$

where $\xi \in (a, b)$.

It is easy to verify that

$$\begin{aligned} f(x) - T(x) &= \frac{E(x)}{Q(x)} \\ &= \frac{\omega(x)E^{(n+1)}(\xi)}{Q(x)(n+1)!} \\ &= \frac{\omega(x)}{Q(x)} \frac{[f(x)Q(x) - P(x)]^{(n+1)}_{x=\xi}}{(n+1)!}. \end{aligned}$$

The proof is completed. □

2.5. Numerical examples

In this section, we present a simple example to show the flexibility and the effectiveness of our method.

Example 1. Let $X_5 = \{0, 1, 2, 3, 4, 5\}$, $\{f_0, f_1, f_2, f_3, f_4, f_5\} = \{0, 1, 4, -1, -2, 2\}$. According to the block-based Thiele-like blending rational interpolation method illustrated in the preceding sections, one can yield the following schemes for interpolants.

Scheme 1: From case 1 in the preceding Section 2.3 and considering that the whole set X_5 is a unique subset, we have

$$\begin{aligned} T(x) &= x + x(x-1) - \frac{5}{3}x(x-1)(x-2) \\ &\quad + \frac{11}{12}x(x-1)(x-2)(x-3) - \frac{33}{120}x(x-1)(x-2)(x-3)(x-4) \\ &= -\frac{463}{30}x + \frac{179}{6}x^2 - \frac{403}{24}x^3 + \frac{11}{3}x^4 - \frac{11}{40}x^5. \end{aligned}$$

Scheme 2: We divide X_5 into $X_5^0 = \{0, 1, 2\}$ and $X_5^1 = \{3, 4, 5\}$.

Let both $I_0(x)$ and $I_1(x)$ be Newton interpolating polynomials; then we have

$$\begin{aligned} T(x) &= x + x(x-1) + \frac{x(x-1)(x-2)}{-\frac{3}{5} - \frac{11}{15}(x-3) - \frac{187}{690}(x-3)(x-4)} \\ &= \frac{-1380x + 3210x^2 - 1493x^3 + 187x^4}{1140 - 803x + 187x^2}. \end{aligned}$$

Scheme 3: We divide X_5 into $X_5^0 = \{0, 1, 2\}$ and $X_5^1 = \{3, 4, 5\}$.

Let $I_0(x)$ be a Newton interpolating polynomial and $I_1(x)$ be a Thiele-type interpolating continued fraction; then we have

$$\begin{aligned} T(x) &= x + x(x-1) + \frac{x(x-1)(x-2)}{-\frac{3}{5} + \frac{x-3}{-\frac{15}{11} + \frac{85(x-4)}{231}}} \\ &= \frac{-262x + 367x^2 - 146x^3 + 17x^4}{-60 + 36x}. \end{aligned}$$

Scheme 4: We divide X_5 into $X_5^0 = \{3, 4, 5\}$ and $X_5^1 = \{0, 1, 2\}$.

Let $I_0(x)$ be a Thiele-type interpolating continued fraction and $I_1(x)$ be a Newton interpolating polynomial; then we have

$$\begin{aligned} T(x) &= -1 + \frac{x-3}{-1 + \frac{5(x-4)}{3}} + \frac{(x-3)(x-4)(x-5)}{-\frac{690}{7} + \frac{2946}{35}x - \frac{77067}{2170}x(x-1)} \\ &= \frac{1067096x - 489506x^2 - 25976x^3 + 10850x^4}{4919700 - 7043037x + 3071136x^2 - 385335x^3}. \end{aligned}$$

Scheme 5: We divide X_5 into $X_5^0 = \{3, 4, 5\}$ and $X_5^1 = \{0, 1, 2\}$.

Let both $I_0(x)$ and $I_1(x)$ be Thiele-type interpolating continued fractions; then we have

$$
\begin{aligned}
T(x) &= -1 + \frac{x-3}{-1+\frac{5(x-4)}{3}} + \frac{(x-3)(x-4)(x-5)}{-\frac{690}{7}+\frac{x}{\frac{35}{2946}+\frac{59941(x-1)}{6912298}}} \\
&= \frac{30220042x - 31479367x^2 + 12338081x^3 - 2084657x^4 + 128445x^5}{21551460 - 14579976x + 2151060x^2}.
\end{aligned}
$$

Scheme 6: From case 2 in the preceding Section 2.3 and considering the whole set X_5 as a unique subset, we have

$$
\begin{aligned}
T(x) &= \frac{x|}{|1} + \frac{x-1|}{|-2} + \frac{x-2|}{|\frac{2}{3}} + \frac{x-3|}{|\frac{3}{4}} + \frac{x-4|}{|-\frac{4}{339}} \\
&= \frac{1502x - 827x^2 + 113x^3}{1800 - 1199x + 187x^2}.
\end{aligned}
$$

3. Multivariate case

3.1. Basic idea

The block-based Thiele-like blending rational interpolation method can be generalized to the multivariate case.

Given a set of two-dimensional points $\Pi_{mn} = \{(x_i, y_j) \mid i = 0, 1, \ldots, m;\ j = 0, 1, \ldots, n\}$, suppose that $f(x, y)$ is defined on $D \supset \Pi_{mn}$.

We divide Π_{mn} into $(u+1) \times (v+1)$ subsets:

$\Pi_{mn}^{st} = \{(x_i, y_j) \mid c_s \leqslant i \leqslant d_s;\ h_t \leqslant j \leqslant r_t\}.(s = 0, 1, \ldots, u;\ t = 0, 1, \ldots, v)$.

Let us consider the following function with a block-based bivariate Thiele-like form:

$$T(x, y) = Z_0(x, y) + \frac{\omega_0(x)|}{|Z_1(x, y)} + \cdots + \frac{\omega_{u-1}(x)|}{|Z_u(x, y)}, \tag{16}$$

where for $s = 0, 1, \ldots, u$

$$Z_s(x, y) = I_{s0}(x, y) + \frac{\omega_0(y)|}{|I_{s1}(x, y)} + \cdots + \frac{\omega_{v-1}(y)|}{|I_{sv}(x, y)}, \tag{17}$$

the polynomials

$$\omega_s(x) = \prod_{i=c_s}^{d_s} (x - x_i), \quad s = 0, 1, \ldots, u-1, \tag{18}$$

$$\omega_t^*(y) = \prod_{j=h_t}^{r_t} (y - y_j), \quad t = 0, 1, \ldots, v-1, \tag{19}$$

and the $I_{st}(x, y)$ $(s = 0, 1, \ldots, u; t = 0, 1, \ldots, v)$ are bivariate polynomials or rational interpolants on the subsets Π_{mn}^{st}.

To obtain a block-based bivariate Thiele-like blending rational interpolant on the whole set Π_{mn}, the $I_{st}(x, y)$ $(s = 0, 1, \ldots, u; t = 0, 1, \ldots, v)$ have to be determined so that function (16) satisfies:

$$T(x_i, y_j) = f(x_i, y_j) = f_{ij}, \quad i = 0, 1, \ldots, m; \quad j = 0, 1, \ldots, n. \tag{20}$$

3.2. Block-based bivariate partial inverse differences

Let $\Pi_{mn} \subset D \subset R^2$, and let $f(x, y)$ be a real function defined on D such that

$$f(x_i, y_j) = f_{ij}, \quad i = 0, 1, \ldots, m; \quad j = 0, 1, \ldots, n. \tag{21}$$

We introduce the following notations:

$$f_{ij}^{00} = f_{ij}, \quad i = 0, 1, \ldots, m; \quad j = 0, 1, \ldots, n \tag{22}$$

for $t = 1, 2, \ldots, v$

$$f_{ij}^{0t} = \frac{\omega_{t-1}^*(y_j)}{f_{ij}^{0,t-1} - I_{0,t-1}(x_i, y_j)}, \quad (j = h_t, h_t + 1, \ldots, n; i = 0, 1, \ldots, m), \tag{23}$$

where $I_{0t}(x, y)$ $(t=0, 1, \ldots, v)$ are bivariate polynomials or rational interpolants on subsets Π_{mn}^{0t}, namely

$$I_{0t}(x_i, y_j) = f_{ij}^{0t}, \quad (c_0 \leqslant i \leqslant d_0, h_t \leqslant j \leqslant r_t, t = 0, 1, \ldots, v) \tag{24}$$

and for $s = 1, 2, \ldots, u$

$$f_{ij}^{s0} = \frac{\omega_{s-1}(x_i)}{f_{ij}^{s-1,0} - Z_{s-1}(x_i, y_j)}, \quad (i = c_s, c_s + 1, \ldots, m; j = 0, 1, \ldots, n), \tag{25}$$

when $t = 1, 2, \ldots, v$

$$f_{ij}^{st} = \frac{\omega_{t-1}^*(y_j)}{f_{ij}^{s,t-1} - I_{s,t-1}(x_i, y_j)} \quad (j = h_t, h_t + 1, \ldots, n; i = c_s, c_s + 1, \ldots, m), \tag{26}$$

where $I_{st}(x, y)$ $(s = 1, 2, \ldots, u; t = 0, 1, \ldots, v)$ are bivariate polynomials or rational interpolants on subsets Π_{mn}^{st}, namely

$$I_{st}(x_i, y_j) = f_{ij}^{st}, \quad (c_s \leqslant i \leqslant d_s, h_t \leqslant j \leqslant r_t; s = 1, 2, \ldots, u; t = 0, 1, \ldots, v). \tag{27}$$

If all f_{ij}^{st} exist, then f_{ij}^{st} are called the (s, t)th block-based bivariate partial inverse differences for function $f(x, y)$.

Theorem 3. *Let*

$$Z_{sv}(x, y) = I_{sv}(x, y), (s = 0, 1, \ldots, u) \tag{28}$$

$$Z_{st}(x,y) = I_{st}(x,y) + \frac{\omega_t^*(y)|}{|I_{s,t+1}(x,y)} + \cdots + \frac{\omega_{v-1}^*(y)|}{|I_{sv}(x,y)}, (s = 0,1,\ldots,u; \; t = 0,1,\ldots,v-1) \tag{29}$$

and

$$T_u(x, y) = Z_u(x, y), \tag{30}$$

$$T_s(x,y) = Z_s(x,y) + \frac{\omega_s(x)|}{|Z_{s+1}(x,y)} + \cdots + \frac{\omega_{u-1}(x)|}{|Z_u(x,y)} \quad (s = 0,1,\ldots,u-1). \tag{31}$$

If all the block-based bivariate partial inverse differences f_{ij}^{st} $(i = c_s, c_s + 1, \ldots, m;\ j = h_t, h_t + 1, \ldots, n;\ s = 0, 1, \ldots, u;\ t = 0, 1, \ldots, v)$ *exist*, (24) *and* (27) *hold, and*

$$Z_{s,t+1}(x, y_j) \neq 0, \quad (s = 0, 1, \ldots, u;\ t = 0, 1, \ldots, v-1;\ j = h_t, h_t + 1, \ldots, r_t) \tag{32}$$

$$T_{s+1}(x_i, y) \neq 0, \quad (s = 0, 1, \ldots, u;\ i = c_s, c_s + 1, \ldots, d_s); \tag{33}$$

then

$$T(x_i, y_j) = f_{ij}, \quad i = 0, 1, \ldots, m; \quad j = 0, 1, \ldots, n.$$

Proof. Let $c_s \leqslant i \leqslant d_s$, and $h_t \leqslant j \leqslant r_t$. From (22) $\sim$ (27), (32) and (33), we have

$$T(x_i,y_j) = Z_0(x_i,y_j) + \frac{\omega_0(x_i)|}{|Z_1(x_i,y_j)} + \cdots + \frac{\omega_{s-1}(x_i)|}{|Z_s(x_i,y_j)},$$

and

$$\begin{aligned} Z_s(x_i,y_j) &= I_{s0}(x_i,y_j) + \frac{\omega_0^*(y_j)|}{|I_{s1}(x_i,y_j)} + \cdots + \frac{\omega_{t-1}^*(y_j)|}{|I_{st}(x_i,y_j)} \\ &= I_{s0}(x_i,y_j) + \frac{\omega_0^*(y_j)|}{|I_{s1}(x_i,y_j)} + \cdots + \frac{\omega_{t-1}^*(y_j)|}{|f_{ij}^{st}} \\ &= I_{s0}(x_i,y_j) + \frac{\omega_0^*(y_j)|}{|I_{s1}(x_i,y_j)} + \cdots + \frac{\omega_{t-3}^*(y_j)|}{|I_{s,t-2}} + \frac{\omega_{t-2}^*(y_j)|}{|f_{ij}^{s,t-1}} \\ &= \cdots = f_{ij}^{s0}. \end{aligned}$$

It is easy to verify that

$$
\begin{aligned}
T(x_i, y_j) &= Z_0(x_i, y_j) + \frac{\omega_0(x_i)}{|Z_1(x_i, y_j)} + \cdots + \frac{\omega_{s-1}(x_i)}{|Z_s(x_i, y_j)} \\
&= Z_0(x_i, y_j) + \frac{\omega_0(x_i)}{|Z_1(x_i, y_j)} + \cdots + \frac{\omega_{s-1}(x_i)}{|f_{ij}^{s0}} \\
&= Z_0(x_i, y_j) + \frac{\omega_0(x_i)}{|Z_1(x_i, y_j)} + \cdots + \frac{\omega_{s-3}(x_i)}{|Z_{s-2}(x_i, y_j)} + \frac{\omega_{s-2}(x_i)}{|f_{ij}^{s-1,0}} \\
&= \cdots = f_{ij}^{00} = f_{ij}.
\end{aligned}
$$

The proof is completed. □

3.3. Error estimation

We now turn to a discussion of the error in the approximation of a function $f(x, y)$ by its block-based bivariate Thiele-like blending rational interpolants. It is easy to verify the following Theorem 4 in terms of a bivariate Newton interpolation formula (see [11]).

Theorem 4. *Suppose $D = [a, b] \times [c, d]$ is a rectangular domain containing Π_{mn} and $f(x, y) \in C^{(m+n+2)}(D)$. Let*

$$
T(x, y) = Z_0(x, y) + \frac{\omega_0(x)}{|Z_1(x, y)} + \cdots + \frac{\omega_{u-1}(x)}{|Z_u(x, y)} = \frac{P(x, y)}{Q(x, y)}
$$

be a block-based bivariate Thiele-like blending rational interpolant on Π_{mn}; then $\forall (x, y) \in D$, we have

$$
\begin{aligned}
f(x, y) - T(x, y) = {} & \frac{\omega(x)}{Q(x, y)} \cdot \frac{\frac{\partial^{m+1}}{\partial x^{m+1}}[fQ - P]_{x=\xi}}{(m+1)!} \\
& + \frac{\omega^*(y)}{Q(x, y)} \cdot \frac{\frac{\partial^{n+1}}{\partial y^{n+1}}[fQ - P]_{y=\eta}}{(n+1)!} - \frac{\omega(x)\omega^*(y)}{Q(x, y)} \cdot \frac{\frac{\partial^{n+m+2}}{\partial x^{m+1}\partial y^{n+1}}[fQ - P]_{x=\overline{\xi}, y=\overline{\eta}}}{(m+1)!(n+1)!},
\end{aligned}
$$

with $\xi, \overline{\xi} \in (a, b)$ and $\eta, \overline{\eta} \in (c, d)$, where

$$\omega(x) = (x - x_0)(x - x_1) \cdots (x - x_m),$$

$$\omega^*(y) = (y - y_0)(y - y_1) \cdots (y - y_n).$$

3.4. Numerical examples

In this section, we present a simple example to show how the algorithms are implemented and how flexible our method is.

Example 2. Suppose the interpolating points and the prescribed values of $f(x, y)$ at the support abscissae (x_i, y_j) are given in the following table:

	$y_0 = 0$	$y_1 = 1$	$y_2 = 2$
$x_0 = 0$	2	6	24
$x_1 = -1$	12	6	12
$x_2 = -2$	0	4	−2

For convenience, we merely present a scheme. Let

$$c_0 = 0, \quad d_0 = 1, \quad c_1 = d_1 = 2; \quad h_0 = 0, \quad r_0 = 1, \quad h_1 = r_1 = 2,$$

which means that Π_{22} is divided into the following 4 subsets Π_{22}^{00}, Π_{22}^{01}, Π_{22}^{10} and Π_{22}^{11}:

$$\begin{array}{ccc} (0,0) & (0,1) & (0,2) \\ (-1,0) & (-1,1) & (-1,2) \\ \\ (-2,0) & (-2,1) & (-2,2) \end{array}$$

Suppose $I_{00}(x, y)$ is a bivariate Newton interpolating polynomial $P_{00}(x, y)$ on the subset Π_{22}^{00}; then we have

$$P_{00}(x, y) = 2 - 10x + 4y + 10xy.$$

By (23), we have

$$f_{02}^{01} = \frac{1}{7}, \quad f_{12}^{01} = \frac{1}{6}, \quad f_{22}^{01} = \frac{1}{4}.$$

Suppose $I_{01}(x, y)$ is a bivariate Newton interpolating polynomial $P_{01}(x, y)$ on the subset Π_{22}^{01}; then we have

$$P_{01}(x, y) = \frac{1}{7} - \frac{x}{42}.$$

By (17), we have

$$Z_0(x, y) = 2 - 10x + 4y + 10xy + \frac{y(y-1)}{\frac{1}{7} - \frac{x}{42}}.$$

It follows from (25) that

$$f_{20}^{10} = -\frac{1}{11}, \quad f_{21}^{10} = -1, \quad f_{22}^{10} = -\frac{4}{5}.$$

Suppose $I_{10}(x, y)$ is a bivariate Newton interpolating polynomial $P_{10}(x, y)$ on the subset Π_{22}^{10}; then we have

$$P_{10}(x, y) = -\frac{1}{11} - \frac{10y}{11}.$$

It follows from (26) that

$$f_{22}^{11} = \frac{110}{61}.$$

Suppose $I_{11}(x, y)$ is a bivariate Newton interpolating polynomial $P_{11}(x, y)$ on the subset Π_{22}^{11}; then we have

$$P_{11}(x, y) = \frac{110}{61}.$$

From (16) and (17), we finally obtain

$$T(x, y) = 2 - 10x + 4y + 10xy + \frac{y(y-1)}{\frac{1}{7} - \frac{x}{42}} + \frac{x(x+1)}{-\frac{1}{11} - \frac{10y}{11} + \frac{y(y-1)}{\frac{110}{61}}}.$$

4. Conclusion and future work

This paper presents a new kind of block-based Thiele-like blending rational interpolants which can be obtained by Thiele's method. Clearly, our method provides us with many flexible interpolation schemes for choices which include the classical Thiele-type continued fraction interpolant as its special case. We give a brief discussion of a block-based Thiele-like blending rational interpolation algorithm and error estimation. A bivariate analogy is also discussed. Through numerical examples, we show the flexibility and effectiveness of the method. Our future work will be focused on the following aspects:

- How to divide the set X_n and how to choose an interpolation method on every subset to obtain a better approximation.
- How to generalize the above block-based interpolation to other formations to obtain a better approximation.
- Applications in image processing.

We conclude this paper by pointing out that it is not difficult to generalize the block-based Thiele-like blending rational interpolation method to a vector-valued case or a matrix-valued case [4,10,12].

References

[1] C. Chaffy, Fractions continues a deux variables et calcul formel, Technical Report, IMAG, 1986.
[2] A. Cuyt, B. Verdonk, Multivariate rational interpolation, Computing 34 (1985) 41–61.
[3] A. Cuyt, B. Verdonk, A review of branched continued fraction theory for the construction of multivariate rational approximants, Appl. Numer. Math. 4 (1988) 263–271.
[4] P.R. Graves-Morris, Vector valued rational interpolants I, Numer. Math. 42 (1983) 331–348.
[5] W. Siemaszko, Thiele-type branched continued fractions for two variable functions, J. Comput. Appl. Math. 9 (1983) 137–153.
[6] J. Stoer, R. Bulirsch, Introduction to Numerical Analysis, second ed., Springer, Berlin, 1992.
[7] J. Tan, Bivariate blending rational interpolants, Approx. Theory Appl. 15 (2) (1999) 74–83.
[8] J. Tan, Y. Fang, Newton-Thiele's rational interpolants, Numer. Algorithms 24 (2000) 141–157.
[9] J. Tan, P. Jiang, A Neville-like method via continued fractions, J. Comput. Appl. Math. 163 (2004) 219–232.

[10] J. Tan, G. Zhu, General framework for vector-valued interpolants, in: Z. Shi (Ed.), Proceedings of the Third China–Japan Seminar on Numerical Mathematics, Science Press, Beijing/New York, 1998, pp. 273–278.
[11] R.H. Wang, Numerical Approximation, Higher Education Press, Beijing, 1999.
[12] G. Zhu, J. Tan, A note on matrix-valued rational interpolants, J. Comput. Appl. Math. 110 (1999) 129–140.